MTH 406: Differential geometry of curves and surfaces

Homework II

Problems for practice

1. Prove the assertion in 1.6(ix) of the Lesson Plan.

2. Show that if γ is a unit-speed curve, then
$$\dot{n}(s) = -\kappa_{\pm}(s)T(s).$$

3. Let γ be a unit-speed space curve. Show that the knowledge of binormal vector $b(s)$, with non-zero torsion, at every point $\gamma(s)$, determines the curvature $\kappa(s)$ and $|\tau(s)|$ of γ.

4. Let γ be a unit-speed space curve. The plane determined by the $T(s)$ and $\eta(s)$ is called the *osculating plane* at $\gamma(s)$.

   (a) Show that osculatins plane at $\gamma(s)$ is the limit position of the plane passing through $\gamma(s)$, $\gamma(s+h_1)$, and $\gamma(s+h_2)$, as $h_1, h_2 \to \infty$.

   (b) Show that the curvature $\kappa(s)$ of γ at $\gamma(s)$ is the curvature of the plane curve $(\pi \circ \gamma)(s)$, where π is the normal projection of γ over the osculating plane at $\gamma(s)$.

5. If a unit-speed simple closed plane curve $\gamma : [0,k] \to \mathbb{R}^2$ is contained in a disk of radius r, then prove that there exists a point $\gamma(s)$ on the curve γ such that $|\kappa(s)| \geq \frac{1}{r}$.

6. Let $\gamma : \mathbb{R} \to \mathbb{R}^2$ be a plane curve. Assume that γ does not pass through the origins, and that $\lim_{t\to\pm\infty} |\gamma(t)| = \infty$.

   (a) Prove that there exists $t_0 \in \mathbb{R}$ such that $|\gamma(t_0)| \leq |\gamma(t)|$, for all $t \in \mathbb{R}$.

   (b) Show, by an example that (a) does not hold, if one does not assume that $\lim_{t\to\pm\infty} |\gamma(t)| = \infty$.

7. Let $\gamma : [0,k] \to \mathbb{R}^2$ be a simple closed plane curve of period k such its curvature $\kappa(s)$ satisfies $0 < \kappa(s) \leq c$, for some constant c, at every point $\gamma(s)$. Show that
$$\ell(\gamma) \geq 2\pi|k|/c.$$